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CLASSIFICATION OF SLUMBILICAL SUBMANIFOLDS IN COMPLEX SPACE FORMS

Dedicated to Professor Shoshichi Kobayashi on the occasion of his seventieth birthday

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1. Introduction

Let M be an n-dimensional Riemannian manifold isometrically immersed in a Kählerian manifold $(\tilde{M}, g, J)$ endowed with Kähler metric g and almost complex structure J. For each vector X tangent to M, we put

$$JX = PX + FX, \tag{1.1}$$

where PX and FX are the tangential and normal components of JX. Then P is an endomorphism of the tangent bundle TM. For any nonzero vector X tangent to M at a point p, the angle $\theta(X)$, $0 \leq \theta(X) \leq \pi/2$, between JX and the tangent space T_pM is called the *Wirtinger angle* of X. The submanifold M is called *slant* if its Wirtinger angle θ is constant, i.e., $\theta(X)$ is independent of the choice of the X in the tangent bundle TM. The Wirtinger angle θ of a slant immersion is called the *slant angle*. A slant submanifold with slant angle θ is simply called *θ-slant*. Slant submanifolds of a Kählerian manifold are characterized by the condition $P^2 = cI$ for some real number $c \in [-1, 0]$. Complex and totally real immersions are slant immersions with slant angle 0 and $\pi/2$, respectively (cf. [4, 10]). A slant immersion is called *proper slant* if it is neither complex nor totally real. A proper slant submanifold is called *Kählerian slant* if its canonical endomorphism P is parallel.

From J-action point of views, slant submanifolds are the simplest and the most natural submanifolds of a Kählerian manifold. Slant submanifolds arise naturally and play some important roles in the studies of submanifolds of Kählerian manifolds. For example, K. Kenmotsu and D. Zhou proved in [9] that every surface in a complex space form $\tilde{M}^2(4c)$ is proper slant if it has constant curvature and nonzero parallel mean curvature vector.

When M is an oriented surface in a Kählerian manifold $\tilde{M}$, one also has the notion of *Kähler angle* α defined by $\alpha = \cos^{-1}(\langle JX, Y\rangle) \in [0, \pi]$, where $\{X, Y\}$ is a local positive orthonormal frame field on M. The Kähler angle α and the Wirtinger angle θ of an oriented surface M are related by $\theta(p) = \min\{\alpha(p), \pi - \alpha(a)\}$. In this sense, an oriented surface in a Kählerian manifold is slant if and only if it has constant Kähler angle.

Let $x : M \to \tilde{M}^m$ be an isometric immersion from a Riemannian manifold into a Kählerian m-manifold. We denote by h and A the second fundamental form and the shape operator of the immersion. And by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively.

The Gauss and Weingarten formulas of M in $\tilde{M}$ are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{1.3}$$

where X, Y are tangent to M and ξ is normal to M. The second fundamental form h and the shape operator A are related by $\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle$. The mean curvature vector H of the immersion is defined by $H = (1/n)\,\mathrm{trace}\,h$, where $\{e_1, \dots, e_n\}$ is a local orthonormal frame field of the tangent bundle TM.

A nonminimal submanifold M of a Riemannian manifold is called *totally umbilical* (or simply *umbilical*) if $h(X, Y) = g(X, Y)H$ for X, Y tangent to M. Clearly, umbilical submanifolds M are the simplest submanifolds which are pseudo-umbilical, i.e., the shape operator of M at H satisfies the condition:

$$A_H X = \mu X \tag{1.4}$$

for any $X \in TM$, where $\mu = g(H, H)$. It is well-known that a umbilical submanifold of a Euclidean space is nothing but an open portion of an ordinary sphere. Umbilical submanifolds (if they exist) are the simplest submanifolds next to totally geodesic ones in Riemannian manifolds from extrinsic point of views. However, since the shape operator of every proper slant surface and also every Kählerian slant submanifold of a Kählerian manifold must satisfy another condition:

$$A_{FX} Y = A_{FY} X, \tag{1.5}$$

for any X, Y tangent to M, there do not exist umbilical Kählerian slant submanifold in a Kählerian n-manifold. For these reasons, it is natural to study the simplest slant submanifolds which satisfy conditions (1.4) and (1.5). We call such submanifolds *slant umbilical submanifolds*, or simply *slumbilical submanifolds.* In some sense, slumbilical submanifolds play the role of umbilical submanifolds of Euclidean space in the family of slant submanifolds. In terms of second fundamental form, an n-dimensional submanifold in a Kählerian manifold is a slumbilical submanifold with slant angle $\theta \in (0, \pi/2]$ if its second fundamental form satisfies

$$\begin{aligned} &h(e_1, e_1) = h(e_2, e_2) = \cdots = h(e_n, e_n) = \lambda e_{1^*}, \\ &h(e_1, e_j) = \lambda e_{j^*}, \quad h(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \dots, n \end{aligned} \tag{1.6}$$

for some function λ with respect to some orthonormal frame field $e_1, \ldots, e_n$, where $e_{1^*} = \csc\theta F e_1, \ldots, e_{n^*} = \csc\theta F e_n$.

The purpose of this article is to obtain the complete classification of slumbilical submanifolds in complex space forms. Our classification theorem (Theorem 4.1) states that there exist twelve families of slumbilical submanifolds in complex space forms with slant angle $\theta \in (0, \pi/2]$. Conversely, every slumbilical submanifold in a complex space form is given by one of these twelve families.

2. Basic formulas and lemmas

Let $\tilde{M}^m(4c)$ denote a Kählerian m-manifold with constant holomorphic sectional curvature $4c$. Such Kählerian manifolds are called *complex space forms*. It is known that the universal covering of a complete complex space form $\tilde{M}^m(4c)$ is the complex projective m-space $CP^m(4c)$, the complex Euclidean n-space $\mathbf{C}^m$, or the complex hyperbolic space $CH^m(4c)$, according to $c > 0$, $c = 0$, or $c < 0$.

Let $x : M \to \tilde{M}^m(4c)$ be an isometric immersion of a Riemannian n-manifold into $\tilde{M}^m(4c)$. Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m(4c)$, respectively. We denote by $\langle\ ,\ \rangle$ the inner product for M as well as for $\tilde{M}^m(4c)$. The Riemann curvature tensor of $\tilde{M}^m(4c)$ satisfies

$$\begin{aligned}\tilde{R}(X, Y; Z, W) = c\{&\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle + \langle JX, W\rangle\langle JY, Z\rangle \\ &- \langle JX, Z\rangle\langle JY, W\rangle + 2\langle X, JY\rangle\langle JZ, W\rangle\}.\end{aligned} \tag{2.1}$$

The well-known *equation of Gauss* is given by

$$\begin{aligned}\tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) &+ \langle h(X, Z), h(Y, W)\rangle \\ &- \langle h(X, W), h(Y, Z)\rangle,\end{aligned} \tag{2.2}$$

for X, Y, Z, W tangent to M and ξ, η normal to M.

For the second fundamental form h, we define its covariant derivative $\bar{\nabla} h$ with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.3}$$

The *equation of Codazzi* is

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{2.4}$$

where $(\tilde{R}(X, Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X, Y)Z$.

For an endomorphism Q on the tangent bundle of the submanifold, we define its covariant derivative ∇Q by $(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y)$. For any vector X tangent to M and each vector ξ normal to M, we put

$$JX = PX + FX, \quad J\xi = t\xi + f\xi \tag{2.5}$$

where PX and FX (respectively, $t\xi$ and $f\xi$) denote the tangential and normal components of JX (respectively, $J\xi$). Suppose M is θ-slant in $\tilde{M}^n(4c)$, then we have [1]

$$P^2 = -(\cos^2\theta)I, \quad \langle PX, Y\rangle + \langle X, PY\rangle = 0, \tag{2.6}$$

$$(\nabla_X P)Y = th(X,Y) + A_{FY}X, \tag{2.7}$$

$$D_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY), \tag{2.8}$$

where I is the identity map.

For an n-dimensional slant submanifold M in $\tilde{M}^m(4c)$ with slant angle $\theta \neq 0$, $F(T_pM)$ is an n-dimensional subspace of the normal space $T_p^{\perp}M$. Moreover, the direct sum $T_pM \oplus F(T_pM)$ is invariant under the action of the almost complex structure J. Thus, for each $p \in M$, there exists a complex subspace ν_p of $T_p\tilde{M}^m(4c)$ such that $T_p\tilde{M}^m(4c) = T_pM \oplus F(T_pM) \oplus \nu_p$ as an orthogonal decomposition.

When M is totally real in $\tilde{M}^n(4c)$, we shall choose an orthonormal local frame $e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}, e_{2n+1}, \ldots, e_{2m}$ on M such that $e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n$, where $e_1, \ldots, e_n$ is a local orthonormal frame on M. If M is proper θ-slant in $\tilde{M}^n(4c)$, then n must be even; say $n = 2k$ (cf. [1]). In this case, we shall choose an orthonormal local frame $e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}$, $e_{2n+1}, \ldots, e_{2m}$ on M such that

$$\begin{aligned} e_2 &= (\sec\theta)Pe_1, \ldots, e_{2k} = (\sec\theta)Pe_{2k-1}, \\ e_{1*} &= (\csc\theta)Fe_1, \ldots, e_{2k^*} = (\csc\theta)Fe_{2k}. \end{aligned} \tag{2.9}$$

We call such orthonormal frames *adapted* (*slant*) *frames.*

By direct computation we also have

$$te_{i^*} = -(\sin\theta)e_i, \quad i = 1, \ldots, 2m, \tag{2.10}$$

$$\begin{aligned} fe_{(2j-1)^*} &= -(\cos\theta)e_{(2j)^*}, \quad fe_{(2j)^*} = (\cos\theta)e_{(2j-1)^*}, \\ Pe_{2j} &= -(\cos\theta)e_{2j-1}, \quad j = 1, \ldots, m. \end{aligned} \tag{2.11}$$

For any vector X tangent to M we put

$$\tilde{\nabla}_X e_i = \sum_{j=1}^n \omega_i^j(X)e_j + \sum_{j=1}^n \omega_i^{j^*}(X)e_{j^*}, \tag{2.12}$$

$$\tilde{\nabla}_X e_{i^*} = \sum_{j=1}^n \omega_{i^*}^j(X)e_j + \sum_{j=1}^n \omega_{i^*}^{j^*}(X)e_{j^*}, \quad i, j = 1, \ldots, n. \tag{2.13}$$

Then $\omega_i^j = -\omega_j^i, \omega_{i^*}^{j^*} = -\omega_{j^*}^{i^*}, \omega_{i^*}^j = -\omega_j^{i^*}$. Moreover, we also have

$$\omega_i^{j^*} = \sum_{k=1}^n h_{ik}^{j^*}\omega^k, \quad h_{ik}^{j^*} = \langle h(e_i, e_k), e_{j^*}\rangle, \tag{2.14}$$

where $\omega^1, \ldots, \omega^n$ is the dual frame of $e_1, \ldots, e_n$.

We need the following lemmas.

Lemma 2.1. *Let M be an n-dimensional ($n = 2k$) proper θ-slant submanifold of a Kählerian m-manifold. Then, with respect to an adapted frame, we have*

$$\omega_{(2i-1)^*}^{(2j-1)^*} - \omega_{2i-1}^{2j-1} = \cot\theta\big(\omega_{2i-1}^{(2j)^*} - \omega_{2i}^{(2j-1)^*}\big) \tag{2.15}$$

$$\omega_{(2j)^*}^{(2i-1)^*} - \omega_{2j}^{2i-1} = \cot\theta\big(\omega_{2i-1}^{(2j-1)^*} + \omega_{2i}^{(2j)^*}\big) \tag{2.16}$$

$$\omega_{(2i)^*}^{(2j)^*} - \omega_{2i}^{2j} = \cot\theta\big(\omega_{2i-1}^{(2j)^*} - \omega_{2i}^{(2j-1)^*}\big), \tag{2.17}$$

for any $i, j = 1, \ldots, k$.

Proof. This lemma was proved by taking the derivatives of the following equations:

$$\langle Je_{(2i-1)^*}, e_{(2j-1)^*}\rangle = \langle Je_{(2i)^*}, e_{(2j)^*}\rangle = 0, \quad \langle Je_{(2i)^*}, e_{(2j-1)^*}\rangle = \cos\theta\delta_{ij}$$

and applying (2.9–13). □

Lemma 2.2. *Let M be an n-dimensional proper θ-slant submanifold of a complex space form $\tilde{M}^m(4c)$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}^m(4c)$ satisfies*

$$(\tilde{R}(X,Y)Z))^{\perp} = c\{\langle JY, Z\rangle FX - \langle JX, Z\rangle FY + 2\langle X, JY\rangle FZ\}, \tag{2.18}$$

for X, Y, Z tangent to M, where $(\tilde{R}(X,Y)Z))^{\perp}$ denotes the normal component of $\tilde{R}(X,Y)Z$.

Proof. Follows from the curvature formula (2.1). □

3. Hopf's fibration and totally real submanifolds

We recall Hopf's fibration and its relationship with totally real real submanifolds in complex projective and complex hyperbolic spaces (cf. [11]).

CASE (1). $\tilde{M}^m(4c) = CP^m(4c)$, $c > 0$.

Let

$$S^{2m+1}(c) = \left\{z = (z_1, \ldots, z_{m+1}) \in \mathbf{C}^{m+1} \ : \ \langle z, z\rangle = \frac{1}{c} > 0\right\}$$

be the hypersphere of constant sectional curvature c centered at the origin.

Consider the Hopf fibration:

$$\pi\colon \ S^{2m+1}(c) \to CP^m(4c). \tag{3.1}$$

Then π is a Riemannian submersion; meaning that π_*, restricted to the horizontal space, is an isometry. Note that given $z \in S^{2m+1}(c)$, the horizontal space at z is the orthogonal complement of iz w.r.t. the metric induced on $S^{2m+1}(c)$ from the usual Hermitian Euclidean metric on $\mathbf{C}^{m+1}$. Moreover, given a horizontal vector X, then iX is again horizontal (and tangent to the sphere) and $\pi_*(iX) = J(\pi_*(X))$, where J is the complex structure on $CP^m(4c)$.

Let $\psi\colon\ M \to CP^m(4c)$ be a totally real isometric immersion. Then there exists an isometric covering map $\tau\colon\ \hat{M} \to M$, and a horizontal isometric immersion $f\colon\ \hat{M} \to S^{2m+1}(c)$ such that $\psi(\tau) = \pi(f)$. Hence every totally real immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a horizontal immersion of the same Riemannian manifold. Conversely, let $f\colon\ \hat{M} \to S^{2m+1}(c)$ be a horizontal isometric immersion. Then $\psi = \pi(f)\colon\ M \to CP^m(4c)$ is again an isometric immersion, which is totally real. Under this correspondence, the second fundamental forms h^f and h^ψ of f and ψ satisfy $\pi_* h^f = h^\psi$. Moreover, h^f is horizontal with respect to π. (We shall denote h^f and h^ψ simply by h.).

CASE (2). $\tilde{M}^m(4c) = CH^m(c)$, $c < 0$.

Consider the complex number $(m+1)$-space $\mathbf{C}_1^{m+1}$ endowed with the pseudo-Euclidean metric g_0 given by

$$g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{m+1} dz_j d\bar{z}_j \tag{3.2}$$

Put

$$H_1^{2m+1}(c) = \left\{ z = (z_1, z_2, \dots, z_{m+1}) : \langle z, z \rangle = \frac{1}{c} < 0 \right\}, \tag{3.3}$$

where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbf{C}_1^{m+1}$ induced from g_0. $H_1^{2m+1}(c)$ is known as an anti-de Sitter space-time.

We put

$$T'_z = \left\{ z \in \mathbf{C}^{m+1} : \operatorname{Re}\langle u, z\rangle = \operatorname{Re}\langle u, iz\rangle = 0 \right\}, \quad H_1^1 = \{\lambda \in C : \lambda\bar{\lambda} = 1\}.$$

Then we have an H_1^1-action on $H_1^{2m+1}(c)$, $z \mapsto \lambda z$, and at each point $z \in H_1^{2m+1}(c)$, the vector iz is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $\operatorname{Re} g_0(iz, iz) = 1/c$. The orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^m(4c)$ with constant holomorphic sectional curvature $4c$, with the complex structure J induced from the canonical complex structure J on C_1^{n+1} via the following totally geodesic fibration:

$$\pi : H_1^{2m+1}(c) \to CH^m(4c). \tag{3.4}$$

Just as in Case (1), let $\psi\colon\ M\to\mathbb{C}H^m(4c)$ be a totally real isometric immersion. Then there exists an isometric covering map $\tau\colon\ \hat{M}\to M$, and a horizontal isometric immersion $f\colon\ \hat{M}\to H_1^{2m+1}(c)$ such that $\psi(\tau)=\pi(f)$. Hence every totally real immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a horizonal immersion. Conversely, let $f:\hat{M}\to H_1^{2m+1}(c)$ be a horizontal isometric immersion. Then $\psi=\pi(f)\colon\ M\to\mathbb{C}H^m(4c)$ is again an isometric immersion, which is totally real. Similarly, under this correspondence, the second fundamental forms h^f and h^ψ of f and ψ satisfy $\pi_* h^f=h^\psi$. Moreover, h^f is horizontal with respect to π. (We shall also denote h^f and h^ψ simply by h.)

4. Classification of slumbilical submanifolds

The main result of this article is the following classification theorem.

Theorem 4.1. *Let $z:M\to\tilde{M}^m(4c)$ be an isometric slant immersion from a Riemannian n-manifold M ($n\geq 2$) into a complete simply-connected complex space form $\tilde{M}^m(4c)$ with slant angle $\theta\in(0,\pi/2]$. Then the immersion is slumbilical if and only if one of the following twelve cases occurs*:

(1) *M is an open portion of the Euclidean n-space E^n and M is immersed as an open portion of a slant n-plane in $\mathbf{C}^m$ ($c=0$).*

(2) *M is an open portion of the real projective n-space $RP^n(c)$ of constant curvature $c>0$ and M is immersed as a totally geodesic totally real submanifold in the complex projective n-space $CP^m(4c)$.*

(3) *M is an open portion of the real hyperbolic n-space $RH^n(c)$ of constant curvature $c<0$ and M is immersed as a totally geodesic totally real submanifold in the complex hyperbolic n-space $CH^m(4c)$.*

(4) *$n=2$ and M is an open portion of the Euclidean 2-plane equipped with the flat metric*

$$g=e^{-2y\cot\theta}\left\{dx^2+(ax+b)^2dy^2\right\} \tag{4.1}$$

for some real numbers a,b with $a\neq 0$. Moreover, up to rigid motions of $\mathbf{C}^m$, the immersion is given by

$$\begin{aligned} z(x,y)=\;&\frac{(ax+b)^{1+ia^{-1}\csc\theta}}{a+i\csc\theta}e^{-y\cot\theta}\Bigg(\cos\left(\sqrt{1+a^2}y\right)\\ &+i\frac{a\cos\theta}{\sqrt{1+a^2}}\sin\left(\sqrt{1+a^2}y\right),\ \frac{a\sin\theta+i}{\sqrt{1+a^2}}\sin\left(\sqrt{1+a^2}y\right),0,\dots,0\Bigg).\end{aligned} \tag{4.2}$$

(5) *$n=2$ and M is an open portion of the Euclidean 2-plane with the flat metric*

$$g=e^{-2y\cot\theta}\left\{dx^2+b^2dy^2\right\} \tag{4.3}$$

for some positive number b. Moreover, up to rigid motions of $\mathbf{C}^m$, the immersion is *given by*

$$z(x, y) = b \sin\theta \exp\left\{ib^{-1}x \csc\theta - y \cot\theta\right\}(\cos y, \sin y, 0, \dots, 0). \tag{4.4}$$

(6) *$\theta = \pi/2$ and M is an open portion the warped product of a line and the unit $(n-1)$-sphere $S^{n-1}(1)$ with the warped metric*

$$g = ds^2 + \frac{(ax+b)^2}{1+a^2} g_1 \tag{4.5}$$

for some real numbers a, b with $a \neq 0$, where g_1 is standard metric on $S^{n-1}(1)$. Moreover, up to rigid motions of $\mathbf{C}^m$, the immersion is given by

$$\begin{aligned} z(x, y_1, \dots, y_n) &= \frac{(ax+b)^{1+ia^{-1}}}{a+i}(y_1, \dots, y_n, 0, \dots, 0), \\ y_1^2 + y_2^2 &+ \cdots + y_n^2 = 1. \end{aligned} \tag{4.6}$$

(7) *$\theta = \pi/2$ and M is an open portion the Riemannian product $\mathbf{R} \times S^{n-1}(1/b^2)$ of a line and the $(n-1)$-sphere $S^{n-1}\left(b^{-2}\right)$ of curvature b^{-2}. Moreover, up to rigid motions of $\mathbf{C}^m$, the immersion is given by*

$$\begin{aligned} z(x, u_2, \dots, u_n) &= b\exp\left\{ib^{-1}x\right\}\left(y_1, \dots, y_n, 0, \dots, 0\right), \\ y_1^2 + y_2^2 &+ \cdots + y_n^2 = 1. \end{aligned} \tag{4.7}$$

(8) *$\theta = \pi/2$, $c > 0$, and M is an open portion the warped product of a line and the unit $(n-1)$-sphere $S^{n-1}(1)$ with the warped metric*

$$g = dx^2 + \frac{\cos^2\left(\sqrt{c}x\right)}{b^2+c} g_1 \tag{4.8}$$

for some positive number b. Moreover, up to rigid motions of $CP^m(4c)$, the immersion z is the composition $\pi \circ \phi$, where $\phi : M \to S^{2m+1}(c) \subset \mathbf{C}^{m+1}$ is given by

$$\begin{aligned} &\phi(x, y_1, \dots, y_n) \\ &= \frac{1}{\sqrt{b^2+c}}\left(\frac{ib + \sqrt{c}\sin\left(\sqrt{c}x\right)}{\sqrt{c}}, \left(\sec\left(\sqrt{c}x\right) + \tan\left(\sqrt{c}x\right)\right)^{ib/\sqrt{c}} y_1, \dots, \right. \\ &\qquad \left. \left(\sec\left(\sqrt{c}x\right) + \tan\left(\sqrt{c}x\right)\right)^{ib/\sqrt{c}} y_n, 0, \dots, 0\right), \quad y_1^2 + y_2^2 + \cdots + y_n^2 = 1, \end{aligned} \tag{4.9}$$

and $\pi : S^{2m+1}(c) \to CP^m(4c)$ is the projection of the Hopf fibration.
(9) *$\theta = \pi/2$, $c < 0$, and M is an open portion the warped product of a line and $S^{n-1}(1)$ with the warped metric*

$$g = dx^2 + \frac{1}{b^2 \exp\left\{2\sqrt{-cx}\right\}} g_1 \tag{4.10}$$

for some positive number b. Moreover, up to rigid motions of $CH^m(4c)$*, the immersion* z *is the composition* $\pi \circ \phi$*, where* $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$ *is given by*

$$\begin{aligned}\phi(x, y_1, \ldots, y_n) = \frac{1}{b}\Bigg(&\frac{ib + \sqrt{-c}\exp\left\{-\sqrt{-c}x\right\}}{\sqrt{-c}},\\ &\exp\left\{-\sqrt{-c}x\right\}\exp\left\{i\left(\frac{b}{\sqrt{-c}}\right)\exp\left(\sqrt{-c}x\right)\right\}y_1, \ldots,\\ &\exp\left\{-\sqrt{-c}x\right\}\exp\left\{i\left(\frac{b}{\sqrt{-c}}\right)\exp\left(\sqrt{-c}x\right)\right\}y_n, 0, \ldots, 0\Bigg),\\ &y_1^2 + y_2^2 + \cdots + y_n^2 = 1,\end{aligned} \tag{4.11}$$

and $\pi : H_1^{2m+1}(c) \to CH^m(4c)$ *is the projection of the hyperbolic Hopf fibration.*
(10) $\theta = \pi/2$, $c < 0$, *and* M *is an open portion the warped product of a line and* $S^{n-1}(1)$ *with the warped metric*

$$g = dx^2 + \frac{\cosh^2\left(\sqrt{-c}x\right)}{b^2 + c}g_1 \tag{4.12}$$

for some positive number b. Moreover, up to rigid motions of $CH^m(4c)$*, the immersion* z *is the composition* $\pi \circ \phi$*, where* $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$ *is given by*

$$\begin{aligned}\phi(x, y_1, \ldots, y_n) = \frac{1}{\sqrt{b^2 + c}}\Bigg(&\frac{ib}{\sqrt{-c}} - \sinh\left(\sqrt{-c}x\right),\\ &\cosh\left(\sqrt{-c}x\right)\exp\left\{2i\left(\frac{b}{\sqrt{-c}}\right)\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}y_1, \ldots,\\ &\cosh\left(\sqrt{-c}x\right)\exp\left\{2i\left(\frac{b}{\sqrt{-c}}\right)\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}y_n, 0, \ldots, 0\Bigg),\\ &y_1^2 + y_2^2 + \cdots + y_n^2 = 1.\end{aligned} \tag{4.13}$$

(11) $\theta = \pi/2$, $c < 0$, *and* M *is an open portion the warped product of a line and the Euclidean* $(n-1)$*-space* E^{n-1} *with the warped metric*

$$g = dx^2 + \cosh^2\left(\sqrt{-c}x\right)g_0, \tag{4.14}$$

where g_0 *denotes the standard metric on* E^{n-1}*. Moreover, up to rigid motions of* $CH^m(4c)$*, the immersion* z *is the composition* $\pi \circ \phi$*, where* $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$

is given by

(4.15)

$$\phi(x, y_2, \dots, y_n) = \frac{\cosh\left(\sqrt{-c}x\right)}{2\sqrt{-c}} \exp\left\{2i \tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}$$

$$\times \Bigg(2\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}}{2}x\right)\right) + \operatorname{sech}^2\left(\sqrt{-c}x\right)\left(i + \sinh\left(\sqrt{-c}x\right)\right) - ic\sum_{j=2}^{n} y_j^2 + i,$$

$$2\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}}{2}x\right)\right) + \operatorname{sech}^2\left(\sqrt{-c}x\right)\left(i + \sinh\left(\sqrt{-c}x\right)\right) - ic\sum_{j=2}^{n} y_j^2 - i,$$

$$2\sqrt{-c}\, y_1, \dots, 2\sqrt{-c}\, y_n, 0, \dots, 0\Bigg).$$

(12) $\theta = \pi/2$, $c < 0$, *and* M *is an open portion the warped product of a line and the real hyperbolic* $(n-1)$*-space* $H^{n-1}(-1)$ *with the warped metric*

$$g = dx^2 - \frac{\cosh^2\left(\sqrt{-c}x\right)}{b^2 + c} g_{-1}, \tag{4.16}$$

where b *is a positive number satisfying* $b^2+c < 0$ *and* g_{-1} *denotes the standard metric on* $H^{n-1}(-1)$ *of constant curvature* -1*. Moreover, up to rigid motions of* $CH^m(4c)$*, the immersion* z *is the composition* $\pi \circ \phi$*, where* $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$ *is given by*

$$\begin{aligned}
&\phi(x, y_1, y_2, \dots, y_n) \\
&\quad = \Bigg(y_1 \frac{\cosh\left(\sqrt{-c}x\right)}{\sqrt{-(b^2+c)}} \exp\left\{2i\left(\frac{b}{\sqrt{-c}}\right)\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}, \dots, \\
&\qquad y_n \frac{\cosh\left(\sqrt{-c}x\right)}{\sqrt{-(b^2+c)}} \exp\left\{2i\left(\frac{b}{\sqrt{-c}}\right)\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}, \\
&\qquad \frac{ib}{\sqrt{c(b^2+c)}} - \frac{\sinh\left(\sqrt{-c}x\right)}{\sqrt{-(b^2+c)}}, 0, \dots, 0\Bigg), \quad y_1^2 - y_2^2 - \cdots - y_n^2 = 1.
\end{aligned} \tag{4.17}$$

When $n = 2$*, the second factor in the product decompositions of* M *mentioned above shall be replaced by a real line.*

Proof. Suppose M is an n-dimensional slumbilical submanifold in $\tilde{M}^m(4c)$ with $m \geq 2$. Then the second fundamental form of M takes the following form:

$$\begin{aligned}
h(e_1, e_1) &= h(e_2, e_2) = \cdots = h(e_n, e_n) = \lambda e_{1^*}, \\
h(e_1, e_j) &= \lambda e_{j^*}, \quad h(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \dots, n
\end{aligned} \tag{4.18}$$

for some function λ with respect to some orthonormal frame field $e_1, \dots, e_n$, where $e_{1^*} = \csc\theta F e_1, \dots, e_{n^*} = \csc\theta F e_n$.

Using (2.14) and (4.18), we have

$$\begin{aligned} &\omega_1^{1^*} = \lambda\omega^1, \quad \omega_1^{j^*} = \omega_j^{1^*} = \lambda\omega^j, \quad \omega_j^{j^*} = \lambda\omega^1, \quad 2 \le j \le n, \\ &\omega_j^{k^*} = 0, \quad 2 \le j \ne k \le n. \end{aligned} \tag{4.19}$$

CASE (α). $\lambda = 0$.

In this case, M is a totally geodesic slant submanifold with slant angle $\theta \in (0, \pi/2]$. When $c = 0$, M is thus an open portion of a Euclidean n-space and M is immersed as an open portion of a slant n-plane in $\mathbf{C}^m$. When $c \ne 0$, M is a totally geodesic totally real submanifold [4]. Moreover, according to [4], M is either an open portion of $RP^n(c)$ or $RH^n(c)$, according to $c > 0$ or $c < 0$, respectively. Hence, when $\lambda = 0$ we obtain statements (1), (2) and (3) of Theorem 4.1.

CASE (β). $\lambda \ne 0$ and $\theta \in (0, \pi/2)$.

In this case, Lemma 2.1 and (4.19) implies

$$\omega_{1^*}^{2^*} = \omega_1^2 - 2\lambda\cot\theta\omega^1, \tag{4.20}$$

$$\omega_{2k-1^*}^{1^*} = \omega_{2k-1}^1 - \lambda\cot\theta\omega^{2k}, \quad k \ge 2, \tag{4.21}$$

$$\omega_{2k^*}^{1^*} = \omega_{2k}^1 + \lambda\cot\theta\omega^{2k-1}, \quad k \ge 2, \tag{4.22}$$

$$\omega_{j^*}^{2^*} = \omega_j^2 - \lambda\cot\theta\omega^j, \quad j \ge 3, \tag{4.23}$$

$$\omega_{l^*}^{j^*} = \omega_l^j, \quad j, l \ge 3, \tag{4.24}$$

From the equation of Codazzi with $X = e_1, Y = Z = e_2$, and using (4.18–24) and Lemma 2.2, we get

$$e_2\lambda = 3\lambda\omega_1^2(e_1) - 2\lambda^2\cot\theta + 3c\sin\theta\cos\theta. \tag{4.25}$$

CASE (β-a). $n \ge 3$.

From the equation of Codazzi with $X = e_1$, $\{Y, Z\} = \{e_2, e_j\}$ for $j \ge 3$, and using (4.18), (4.19) and Lemmas 2.1 and 2.2, we find

$$e_2\lambda = \lambda\omega_1^2(e_1) + 2\epsilon\sin\theta\cos\theta. \tag{4.26}$$

Combining (4.25) and (4.26) we get

$$2\lambda\omega_1^2(e_1) = 2\lambda^2\cot\theta - c\sin\theta\cos\theta. \tag{4.27}$$

From the equation of Codazzi with $X = Z = e_1, Y = e_{2j-1}$ for $j > 1$, and using (4.18–24) and Lemma 2.2, we find

$$\omega_1^{2j}(e_{2j-1}) = -\lambda\cot\theta. \tag{4.28}$$

Similarly, from the equation of Codazzi with $X = e_{2j-1}, Y = e_1, Z = e_{2j}$ for $j > 1$, and using (4.18–24) and Lemma 2.2, we find

$$\lambda\omega_1^{2j}(e_{2j-1}) = c\sin\theta\cos\theta + \lambda^2\cot\theta \tag{4.29}$$

by comparing the coefficients of e_{1^*}. Combining (4.28) and (4.29) yields

$$c\sin\theta\cos\theta + 2\lambda^2\cot\theta = 0. \tag{4.30}$$

Substituting (4.30) into (4.27) yields

$$\omega_1^2(e_1) = 2\lambda\cot\theta. \tag{4.31}$$

On the other hand, from the equation of Codazzi with $X = e_2, Y = Z = e_j$ for $j \geq 3$, and applying (4.18–24) and Lemma 2.2, we find

$$e_2\lambda = 0, \tag{4.32}$$

Combining (4.26) and (4.32) yields

$$\lambda\omega_1^2(e_1) = -2c\sin\theta\cos\theta. \tag{4.33}$$

Equations (4.31) and (4.33) imply

$$c\sin\theta\cos\theta + \lambda^2\cot\theta = 0. \tag{4.34}$$

From (4.30) and (4.34) we obtain $\lambda = 0$ which is a contradiction. Therefore, this case cannot occur.

CASE (β-b). $n = 2$.

In this case, (4.18) reduces to

$$h(e_1, e_1) = h(e_2, e_2) = \lambda e_{1^*}, \quad h(e_1, e_2) = \lambda e_{2^*}. \tag{4.35}$$

From (4.35) and the equation of Codazzi we have

$$e_1\lambda = \lambda\omega_2^1(e_2), \tag{4.36}$$

$$e_2\lambda = \lambda^2\cot\theta + 3c\sin\theta\cos\theta, \tag{4.37}$$

$$\omega_1^2(e_1) = \lambda\cot\theta, \tag{4.38}$$

Moreover, from (4.35–38) and the equation of Gauss, we find

$$e_1e_1(\ln\lambda) - (e_1\ln\lambda)^2 - c = \lambda^2\cot^2\theta - \cot\theta\, e_2\lambda + 3c\cos^2\theta. \tag{4.39}$$

Therefore, by applying (4.37), we get

$$e_1e_1(\ln\lambda) - (e_1\ln\lambda)^2 = c. \tag{4.40}$$

Let μ be a function on M. Then, (4.36) and (4.38) imply that $[\mu e_1, \lambda^{-1}e_2] = 0$ if and only if μ satisfies

$$e_2(\ln\mu) = -\cot\theta. \tag{4.41}$$

Thus, there exists a coordinate chart $\{x, y\}$ such that $\mu e_1 = \partial/\partial x$ and $\lambda^{-1}e_2 = \partial/\partial y$ if and only if $\partial/\partial y(\ln\mu) = -\cot\theta$. Consequently, by putting $\mu = e^{-y\cot\theta}$, we obtain a coordinate chart $\{x, y\}$ such that

$$e_1 = e^{y\cot\theta}\frac{\partial}{\partial x}, \quad e_2 = \lambda\frac{\partial}{\partial y}. \tag{4.42}$$

From (4.42) we know that the metric tensor of M is given by

$$g = e^{-2y\cot\theta}dx^2 + \frac{1}{\lambda^2}dy^2. \tag{4.43}$$

Using (4.37) and (4.42) we obtain

$$\lambda\frac{\partial\lambda}{\partial y} = \lambda^2\cot\theta + 3c\sin\theta\cos\theta. \tag{4.44}$$

Solving (4.44) yields

$$\lambda = \pm\sqrt{\varphi(x)e^{2y\cot\theta} - 3c\sin^2\theta}. \tag{4.45}$$

First, we assume that

$$\lambda = \sqrt{\varphi(x)e^{2y\cot\theta} - 3c\sin^2\theta}. \tag{4.46}$$

Then (4.42) implies

$$\begin{aligned} e_1(\ln\lambda) &= \frac{e^{3y\cot\theta}}{2\lambda^2}\varphi'(x), \\ e_1e_1(\ln\lambda) &= \frac{e^{4y\cot\theta}}{2\lambda^4}\left\{\lambda^2\varphi''(x) - e^{2y\cot\theta}\varphi'(x)^2\right\}. \end{aligned} \tag{4.47}$$

Substituting (4.47) into (4.40) gives

$$\begin{aligned} &\left(2\varphi(x)\,\varphi''(x) - \varphi'(x)^2\right)e^{6y\cot\theta} - \left(6c\sin^2\theta\varphi''(x) + 4\,c\,\varphi(x)^2\right)e^{4y\cot\theta} \\ &\quad +24c^2\varphi(x)\sin^2\theta e^{2y\cot\theta} + 36c^3\sin^4\theta = 0. \end{aligned} \tag{4.48}$$

Since (4.48) holds true on the whole coordinate neighborhood, we obtain $c = 0$. Similarly, we also have $c = 0$ for the case: $\lambda = -\sqrt{\varphi(x)e^{2y\cot\theta} - 3c\sin^2\theta}$. Thus, in both cases, we obtain from (4.45) that

$$\lambda = q(x)e^{y\cot\theta} \tag{4.49}$$

for some function $q(x)$.

From (4.40), (4.42) and (4.49), we find that $q = q(x)$ satisfies the following second order differential equation:

$$qq'' = 2q'^2. \tag{4.50}$$

Solving (4.50) yields $q(x) = 1/(ax + b)$ for some constants a and b. Thus we get

$$\lambda = \frac{e^{y\cot\theta}}{ax + b}. \tag{4.51}$$

From (4.43) and (4.51), we find

$$\begin{aligned} \nabla_{\partial/\partial x}\frac{\partial}{\partial x} &= \frac{\cot\theta}{(ax+b)^2}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial x}\frac{\partial}{\partial y} &= -\cot\theta\frac{\partial}{\partial x} + \frac{a}{ax+b}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial y}\frac{\partial}{\partial y} &= -a(ax+b)\frac{\partial}{\partial x} - \cot\theta\frac{\partial}{\partial y}. \end{aligned} \tag{4.52}$$

On the other hand, using (2.5), (2.12), (4.35) and (4.51), we have

$$\begin{aligned} h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= \frac{\csc\theta}{ax+b}J\left(\frac{\partial}{\partial x}\right) - \frac{\cot\theta}{(ax+b)^2}\frac{\partial}{\partial y}, \\ h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= (\cot\theta)\frac{\partial}{\partial x} + \frac{\csc\theta}{ax+b}J\left(\frac{\partial}{\partial y}\right), \\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) &= (ax+b)\csc\theta J\left(\frac{\partial}{\partial x}\right) - (\cot\theta)\frac{\partial}{\partial y}. \end{aligned} \tag{4.53}$$

Let $z = z(x, y)$ denote the immersion of M into $\mathbf{C}^m$. Then (4.52), (4.53) and the formula of Gauss imply that z satisfies the following system of partial differential equations:

$$\begin{aligned} z_{xx} &= \left(\frac{i\csc\theta}{ax+b}\right)z_x, \\ z_{xy} &= \left(\frac{a + i\csc\theta}{ax+b}\right)z_y, \\ z_{yy} &= (i\csc\theta - a)(ax+b)z_x - 2\cot\theta z_y. \end{aligned} \tag{4.54}$$

Case (β-b-1). $a \neq 0$.

Solving the first equation of (4.54) yields

$$z(x, y) = A(y) + (ax + b)^{1+ia^{-1}\csc\theta} B(y), \tag{4.55}$$

for some $\mathbf{C}^m$-valued functions $A(y)$ and $B(y)$. Substituting (4.55) into the second equation of (4.54) shows that $A(y)$ is a constant vector. Thus, we may choose $A = 0$ by applying a suitable translation on $\mathbf{C}^m$ if necessary. Hence, we get

$$z(x, y) = (ax + b)^{1+ia^{-1}\csc\theta} B(y), \tag{4.56}$$

Substituting (4.56) into the third equation of (4.54) yields

$$B''(y) + 2\cot\theta B'(y) + (a^2 + \csc^2\theta)B(y) = 0.$$

By solving this differential equation and using (4.56), we find

$$\begin{aligned} z(x, y) &= (ax + b)^{1+ia^{-1}\csc\theta} e^{-y\cot\theta} \\ &\quad \times \left(c_1 \cos\left(\sqrt{1 + a^2}\, y\right) + c_2 \sin\left(\sqrt{1 + a^2}\, y\right)\right), \end{aligned} \tag{4.57}$$

where c_1 and c_2 are constant vectors in $\mathbf{C}^m$.

If we choose the following initial conditions:

$$z_x(0, 0) = \left(b^{ia^{-1}\csc\theta}, 0, \ldots, 0\right), \quad z_y(0, 0) = b^{1+ia^{-1}\csc\theta}(i\cos\theta, \sin\theta, 0, \ldots, 0),$$

then (4.57) implies

$$c_1 = \left(\frac{1}{a + i\csc\theta}, 0, \ldots, 0\right), \quad c_2 = \frac{1}{\sqrt{1 + a^2}} \left(\frac{ia\cos\theta}{a + i\csc\theta}, \sin\theta, 0, \ldots, 0\right). \tag{4.58}$$

From (4.57) and (4.58) we obtain (4.2). This gives statement (4) of Theorem 4.1.

Case (β-b-2). $a = 0$.

In this case, (4.54) becomes

$$z_{xx} = \left(\frac{i\csc\theta}{b}\right) z_x, \quad z_{xy} = \left(\frac{i\csc\theta}{b}\right) z_y, \quad z_{yy} = ib\csc\theta z_x - 2\cot\theta z_y. \tag{4.59}$$

Solving the first equation in (4.59) yields

$$z(x, y) = A(y) + \exp\left\{ib^{-1}x\csc\theta\right\} B(y). \tag{4.60}$$

Substituting (4.60) into the second equation of (4.59) shows that A is a constant vector. Without loss of generality, we may choose $A = 0$.

Substituting (4.60) with $A = 0$ into the third equation of (4.59) yields $B'' + 2\cot\theta\, B' + \csc^2\theta\, B = 0$. Hence, we obtain

$$z(x, y) = \exp\left\{ib^{-1}x\csc\theta - y\cot\theta\right\}(c_1\cos(y) + c_2\sin(y)), \tag{4.61}$$

for some constant vectors c_1 and c_2 in $\mathbf{C}^m$.

If we choose the following initial conditions:

$$z_x(0,0) = (i, 0, \dots, 0), \quad z_y(0,0) = (-b\cos\theta, b\sin\theta, 0, \dots, 0),$$

then we obtain $c_1 = (b\sin\theta, 0, \dots, 0)$, $c_2 = (0, b\sin\theta, 0, \dots, 0)$. Thus, we obtain statement (5) of Theorem 4.1 in this case.

CASE (γ). $\lambda \neq 0$ and $\theta = \pi/2$.

In this case, the second fundamental form of M takes the form:

$$\begin{aligned} &h(e_1, e_1) = \cdots = h(e_n, e_n) = \lambda J e_1, \\ &h(e_1, e_j) = \lambda J e_j, \quad h(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \dots, n \end{aligned} \tag{4.62}$$

with respect to some orthonormal frame $e_1, \dots, e_n$.

From (4.62) and the equation of Codazzi we find

$$e_1(\ln\lambda) = \omega_2^1(e_2) = \cdots = \omega_n^1(e_n), \tag{4.63}$$

$$e_j\lambda = 0, \quad \omega_1^j(e_1) = 0, \quad j = 2, \dots, n, \tag{4.64}$$

$$\omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n. \tag{4.65}$$

From (4.63) and Cartan's structure equations, we have

$$e_1e_1(\ln\lambda) - (e_1\ln\lambda)^2 = c. \tag{4.66}$$

Using (4.63), (4.64), and (4.65) we get

$$d\omega^1 = 0, \quad \omega_j^1 = e_1(\ln\lambda)\omega^j, \quad j = 2, \dots, n, \tag{4.67}$$

which implies that the integral curves of e_1 are geodesics in M.

Let $\mathcal{D}$ and $\mathcal{D}^\perp$ denote the distributions spanned by $\{e_1\}$ and $\{e_2, \dots, e_n\}$, respectively. From (4.65) we know that $\mathcal{D}^\perp$ is integrable as well as $\mathcal{D}$ is trivially integrable, since $\mathcal{D}$ is of rank one. Thus, there exist local coordinate systems $\{x, x_2 \dots, x_n\}$ such that $\mathcal{D}$ is spanned by $\partial/\partial x$ and $\mathcal{D}^\perp$ is spanned by $\partial/\partial x_2, \dots, \partial/\partial x_n$. Moreover, since $d\omega^1 = 0$, we may choose $x = x_1$ such that $\omega^1 = dx$ and $e_1 = \partial/\partial x$. From (4.64) it follows that λ is independent of $x_2, \dots, x_n$. Thus, $\lambda = \lambda(x)$.

Using (4.62) and the equation of Codazzi we may obtain as in [2, p. 92] that

$$\langle\nabla_X Y, e_1\rangle = -e_1(\ln\lambda)\langle X, Y\rangle, \tag{4.68}$$

for vector fields X, Y in $\mathcal{D}^\perp$. Thus, the leaves of $\mathcal{D}^\perp$ are totally umbilical hypersurfaces in M with parallel mean curvature vector vector. Hence, $\mathcal{D}^\perp$ is a spherical distribution.

From (4.62), (4.68) and the equation of Gauss, we know that each leaf of $\mathcal{D}^\perp$ is of constant sectional curvature $u(x)$ given by

$$u(x) = c + \lambda^2(x) + (\ln \lambda(x))'^2. \tag{4.69}$$

On the other hand, since the integral curves of $\mathcal{D}$ are geodesics, the distribution $\mathcal{D}$ is auto-parallel. Therefore, by applying a result of Hiepko [8] (see, also [7]), the equations of Gauss and Codazzi, we conclude that M is locally a warped product $I \times_{f(x)} N^{n-1}(\epsilon)$, where $f = 1/\sqrt{u}, 1/\lambda$, or $1/\sqrt{-u}$ and $N(\epsilon)$ is a space of constant curvature ϵ, $\epsilon = 1, 0$, or -1, according to $u > 0, u = 0$, or $u < 0$, respectively. When $n = 2$, the second factor $N^{n-1}(\epsilon)$ in the warped product decomposition shall be replaced by a real line $\mathbf{R}$.

From (4.66) and $e_1 = \partial/\partial x$, we get

$$\lambda\lambda'' = 2\lambda'^2 + c\lambda^2. \tag{4.70}$$

Solving (4.70) yields

$$\lambda'(x)^2 = \alpha\lambda^4(x) - c\lambda^2(x), \tag{4.71}$$

for some constant α. It is easy to verify that the nontrivial solutions of (4.71) are given by

$$\lambda = \begin{cases} (ax+b)^{-1} & \text{if} \quad c = 0, \\ b \operatorname{sech}\left(\sqrt{c}x\right) & \text{if} \quad c > 0, \\ b \exp\left\{\sqrt{-c}x\right\} \ \text{ or } \ b \operatorname{sech}\left(\sqrt{-c}x\right) & \text{if} \quad c < 0, \end{cases} \tag{4.72}$$

where $a^2 + b^2 \neq 0$ when $c = 0$; and $b \neq 0$ when $c \neq 0$.

CASE (γ-a-1). $c = 0$ and $\lambda = (ax + b)^{-1}$ with $a \neq 0$.

In this case, M is a warped product of $\mathbf{R}$ and $S^{n-1}(1)$ with the warped metric given by

$$g = \begin{cases} dx^2 + \dfrac{(ax+b)^2}{1+a^2} dx_2^2 & \text{if} \quad n = 2, \\ dx^2 + \dfrac{(ax+b)^2}{1+a^2} g_1 & \text{if} \quad n \geq 3, \end{cases} \tag{4.73}$$

where

$$g_1 = dx_2^2 + \cos^2 x_2 dx_3^2 + \cdots + \cos^2 x_2 \cdots \cos^2 x_{n-1} dx_n^2 \tag{4.74}$$

is the metric on $S^{n-1}(1)$ with respect to spherical coordinates $\{x_2, \dots, x_n\}$.

From (4.62), (4.73) and the formula of Gauss, we know that the immersion z of M in $\mathbf{C}^m$ satisfies

$$\tag{4.75} \begin{aligned} z_{xx} &= \left(\frac{i}{ax+b}\right) z_x, \\ \tilde{\nabla}_Y z_x &= \left(\frac{a+i}{ax+b}\right) Y, \\ \tilde{\nabla}_Y \tilde{\nabla}_Z z &= \left(\frac{i}{ax+b}\right) \langle Y, Z\rangle\, z_x + \nabla_Y Z, \end{aligned}$$

for Y, Z tangent to the second component $N(\epsilon)$ of the warped decomposition of M.

Solving system (4.75) yields

$$\tag{4.76} \begin{aligned} z = (ax+b)^{1+ia^{-1}} & \Bigg(c_1 \prod_{j=2}^{n} \cos x_j + c_2 \sin x_2 \\ & + c_3 \sin x_3 \cos x_2 + \cdots\cdots + c_n \sin x_n \prod_{j=2}^{n-1} \cos x_j \Bigg) \end{aligned}$$

for some vectors $c_1, \dots, c_n \in \mathbf{C}^m$. Hence, if we choose the initial conditions:

$$\tag{4.77} \begin{aligned} z_x(0, \dots, 0) &= \left((a+i) b^{ia^{-1}}, 0 \dots, 0\right), \\ z_{x_2}(0, \dots, 0) &= \left(0, b^{1+ia^{-1}}, 0, \dots, 0\right), \\ &\ \ \vdots \\ z_{x_n}(0, \dots, 0) &= \left(0, \dots, 0, b^{1+ia^{-1}}, 0, \dots, 0\right), \end{aligned}$$

then we obtain (4.6). This gives statement (6) of Theorem 4.1.

CASE (γ-a-2). $c = 0$ and $\lambda = 1/b$.

In this case, system (4.75) reduces to

$$\tag{4.78} \begin{aligned} & z_{xx} = \left(\frac{i}{b}\right) z_x, \quad \tilde{\nabla}_Y z_x = \left(\frac{i}{b}\right) Y, \\ & \tilde{\nabla}_Y \tilde{\nabla}_Z z = \left(\frac{i}{b}\right) \langle Y, Z\rangle\, z_x + \nabla_Y Z, \end{aligned}$$

By applying an argument similar to case (γ-a-1) we obtain statement (7) of Theorem 4.1 in this case.

CASE (γ-b). $c > 0$ and $\lambda = b \sec\left(\sqrt{c}x\right)$.

In this case, M is a warped product of $\mathbf{R}$ and $S^{n-1}(1)$ with the warped metric given by

$$g = \begin{cases} dx^2 + \dfrac{\cos^2\left(\sqrt{c}\,x\right)}{b^2+c}dx_2^2 & \text{if} \quad n = 2, \\ dx^2 + \dfrac{\cos^2\left(\sqrt{c}\,x\right)}{b^2+c}g_1 & \text{if} \quad n \geq 3. \end{cases} \tag{4.79}$$

From (4.62), (4.79) and the formula of Gauss, we know that the horizontal lift $\phi : M \to S^{2m+1}(c) \subset \mathbf{C}^{m+1}$ of $z : M \to CP^m(4c)$ satisfies

$$\begin{aligned} &\phi_{xx} = ib\sec\left(\sqrt{c}\,x\right)\phi_x - c\phi, \\ &\tilde{\nabla}_Y\phi_x = \left(ib\sec\left(\sqrt{c}\,x\right) - \sqrt{c}\tan\left(\sqrt{c}\,x\right)\right)Y, \\ &\tilde{\nabla}_Y\tilde{\nabla}_Z\phi = \left\{ib\sec\left(\sqrt{c}\,x\right)\phi_x - c\phi\right\}\langle Y, Z\rangle + \nabla_Y Z, \end{aligned} \tag{4.80}$$

for Y, Z tangent to the second component $N(\epsilon)$ of the warped decomposition of M.

Solving the first equation of (4.80) yields

$$\begin{aligned} \phi \;=\; & A(x_2, \ldots, x_n)\left(ib + \sqrt{c}\sin\left(\sqrt{c}\,x\right)\right) \\ & + B(x_2, \ldots, x_n)\cos\left(\sqrt{c}\,x\right)\left(\sec\left(\sqrt{c}\,x\right) + \tan\left(\sqrt{c}\,x\right)\right)^{ib/\sqrt{c}}. \end{aligned} \tag{4.81}$$

The second equation in (4.80) and (4.81) imply $\partial A/\partial x_j = 0$ for $j = 2, \ldots, n$. Thus, A is a constant vector, say c_0, in $\mathbf{C}^{m+1}$.

By applying (4.81) and the last equation of (4.80) with $Y = Z = \partial/\partial x_2$, we obtain

$$\begin{aligned} \phi \;=\; & c_0\left(ib + \sqrt{c}\sin\left(\sqrt{c}\,x\right)\right) + (c_1(x_3, \ldots, x_n)\sin x_2 \\ & + B_1(x_3, \ldots, x_n)\cos x_2)\cos\left(\sqrt{c}\,x\right)\left(\sec\left(\sqrt{c}\,x\right) + \tan\left(\sqrt{c}\,x\right)\right)^{ib/\sqrt{c}}. \end{aligned} \tag{4.82}$$

By applying (4.82) and the third equation of (4.80) with $Y = \partial/\partial x_2$, $Z = \partial/\partial x_j$, $j = 3, \ldots, n$, we conclude that c_1 is a constant vector. Furthermore, by applying (4.82) and the third equation of (4.80) with $Y = Z = \partial/\partial x_3$, we have

$$B_1 = (\sin x_3)c_2(x_4, \ldots, x_n) + (\cos x_3)B_2(x_4, \ldots, x_n). \tag{4.83}$$

By applying (4.82), (4.83) and the third equation of (4.80) with $Y = \partial/\partial x_3$, $Z = \partial/\partial x_j$, $j = 4, \ldots, n$, we also know that c_2 is a constant vector. Continue such process $n-1$ times, we obtain

$$\begin{aligned} \phi = & \, c_0\left(ib + \sqrt{c}\sin\left(\sqrt{c}\,x\right)\right) \\ & + \left(c_1\sin x_2 + c_2\sin x_3\cos x_2 + \cdots + c_{n-1}\sin x_{n-1}\prod_{k=2}^{n-2}\cos x_k + c_n\prod_{k=2}^{n-1}\cos x_k\right) \\ & \times \left(\sec\left(\sqrt{c}\,x\right) + \tan\left(\sqrt{c}\,x\right)\right)^{ib/\sqrt{c}}. \end{aligned} \tag{4.84}$$

By choosing the initial conditions:

$$\begin{aligned}
\phi(0,\dots,0) &= \left(\frac{ib}{\sqrt{c(b^2+c)}}, \frac{1}{\sqrt{b^2+c}}, 0,\dots,0\right),\\
\phi_x(0,\dots,0) &= \left(\frac{\sqrt{c}}{\sqrt{b^2+c}}, \frac{ib}{\sqrt{b^2+c}}, 0\dots,0\right),\\
\phi_{x_2}(0,\dots,0) &= \left(0,0,\frac{1}{\sqrt{b^2+c}}, 0,\dots,0\right),\\
&\vdots\\
\phi_{x_n}(0,\dots,0) &= \left(0,\dots,0,\frac{1}{\sqrt{b^2+c}}, 0,\dots,0\right),
\end{aligned}$$

we obtain (4.9) for ϕ. Thus, we obtain statement (8) in this case.

CASE (γ-c). $c < 0$ and $\lambda = b\exp\left\{\sqrt{-c}x\right\}$.

In this case, M is a warped product of $\mathbf{R}$ and $S^{n-1}(1)$ with the warped metric given by

$$g = \begin{cases} dx^2 + \dfrac{1}{b^2\exp\left\{2\sqrt{-c}x\right\}}dx_2^2 & \text{if} \quad n=2,\\ dx^2 + \dfrac{1}{b^2\exp\left\{2\sqrt{-c}x\right\}}g_1 & \text{if} \quad n\geq 3. \end{cases} \tag{4.85}$$

From (4.62), (4.85) and the formula of Gauss, we know that the horizontal lift $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$ of $z : M \to CH^m(4c)$ satisfies

$$\begin{aligned}
&\phi_{xx} = ib\exp\left\{\sqrt{-c}x\right\}\phi_x - c\phi,\\
&\tilde{\nabla}_Y\phi_x = \left(ib\exp\left\{\sqrt{-c}x\right\} - \sqrt{-c}\,\right)Y,\\
&\tilde{\nabla}_Y\tilde{\nabla}_Z\phi = \{ib\exp\left\{\sqrt{-c}x\right\}\phi_x - c\phi\}\langle Y,Z\rangle + \nabla_Y Z.
\end{aligned}\tag{4.86}$$

Solving system (4.86) as in case (γ-b) yields

$$\begin{aligned}
\phi &= c_0\left(ib + \sqrt{-c}\exp\left\{-\sqrt{-c}x\right\}\right)\\
&+ \left(c_1\sin x_2 + c_2\sin x_3\cos x_2 + \cdots + c_{n-1}\sin x_{n-1}\prod_{k=2}^{n-2}\cos x_k + c_n\prod_{k=2}^{n-1}\cos x_k\right)\\
&\times\exp\left\{-\sqrt{-c}x\right\}\exp\left\{i\left(\frac{b}{\sqrt{-c}}\right)\exp\left\{\sqrt{-c}x\right\}\right\}.
\end{aligned}\tag{4.87}$$

By choosing the initial conditions:

$$\phi(0,\dots,0) = \frac{1}{b}\left(\frac{ib+\sqrt{-c}}{\sqrt{-c}}, \exp\left\{\frac{ib}{\sqrt{-c}}\right\}, 0,\dots,0\right),$$

$$\begin{aligned}
\phi_x(0,\dots,0) &= \frac{1}{b}\left(-\sqrt{-c}, (ib-\sqrt{-c})\exp\left\{\frac{ib}{\sqrt{-c}}\right\}, 0,\dots,0\right),\\
\phi_{x_2}(0,\dots,0) &= \frac{1}{b}\left(0, 0, \exp\left\{\frac{ib}{\sqrt{-c}}\right\}, 0,\dots,0\right),\\
&\vdots\\
\phi_{x_n}(0,\dots,0) &= \frac{1}{b}\left(0,\dots,0, \exp\left\{\frac{ib}{\sqrt{-c}}\right\}, 0,\dots,0\right),
\end{aligned}$$

we obtain (4.11) for ϕ. Thus, we obtain statement (9) in this case.

CASE (γ-d). $c < 0,\ \lambda = b\ \mathrm{sech}\left(\sqrt{c}x\right)$

In this case, we have $u(x) = c + \lambda^2(x) + (\ln\lambda'(x))^2 = (b^2+c)\,\mathrm{sech}^2\left(\sqrt{-c}x\right)$. We divide this case into three subcases.

CASE (γ-d-1). $b^2 + c > 0$.

In this case, M is the warped product of $\mathbf{R}$ and $S^{n-1}(1)$ with the warped metric given by

$$g = \begin{cases} dx^2 + \dfrac{\cosh^2\left(\sqrt{-c}x\right)}{b^2+c}dx_2^2 & \text{if} \quad n = 2,\\[2ex] dx^2 + \dfrac{\cosh^2\left(\sqrt{-c}x\right)}{b^2+c}g_1 & \text{if} \quad n \geq 3. \end{cases} \tag{4.88}$$

From (4.62), (4.88) and the formula of Gauss, we know that the horizontal lift $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$ of $z : M \to CH^m(4c)$ satisfies

$$\begin{aligned}
&\phi_{xx} = ib\ \mathrm{sech}\left(\sqrt{-c}\,x\right)\phi_x - c\phi,\\
&\tilde\nabla_Y\phi_x = \left(ib\ \mathrm{sech}\left(\sqrt{-c}\,x\right) + \sqrt{-c}\tanh\left(\sqrt{-c}\,x\right)\right)Y,\\
&\tilde\nabla_Y\tilde\nabla_Z\phi = \left\{ib\ \mathrm{sech}\left(\sqrt{-c}\,x\right)\phi_x - c\phi\right\}\langle Y, Z\rangle + \nabla_Y Z.
\end{aligned}\tag{4.89}$$

Solving system (4.89) as in case (γ-b) yields

$$\begin{aligned}
\phi = c_0\left(ib - \sqrt{-c}\sinh\left(\sqrt{-c}\,x\right)\right) + \Bigg(&c_1\sin x_2 + c_2\sin x_3\cos x_2 + \cdots\\
&+c_{n-1}\sin x_{n-1}\prod_{k=2}^{n-2}\cos x_k + c_n\prod_{k=2}^{n-1}\cos x_k\Bigg)\\
&\times\cosh\left(\sqrt{-c}\,x\right)\exp\left\{2i\left(\frac{b}{\sqrt{-c}}\right)\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}.
\end{aligned}\tag{4.90}$$

By choosing the initial conditions:

$$\begin{aligned}
\phi(0,\dots,0) &= \frac{1}{\sqrt{b^2+c}}\left(\frac{ib}{\sqrt{-c}},1,0,\dots,0\right),\\
\phi_x(0,\dots,0) &= \frac{1}{\sqrt{b^2+c}}\left(-\sqrt{-c},ib,0,\dots,0\right),\\
\phi_{x_2}(0,\dots,0) &= \frac{1}{\sqrt{b^2+c}}(0,0,1,0,\dots,0),\\
&\ \vdots\\
\phi_{x_n}(0,\dots,0) &= \frac{1}{\sqrt{b^2+c}}(0,\dots,0,1,0,\dots,0),
\end{aligned}$$

we obtain (4.13) for ϕ. Thus, we obtain statement (10) in this case.

CASE (γ-d-2). $b^2+c=0$.

In this case, Hiepko's result implies that M is locally a warped product of a real line and E^{n-1} with warped product metric given by

$$g = dx^2 + b^2\cosh^2\left(\sqrt{-c}\,x\right)\left\{dx_2^2+dx_3^2+\cdots+dx_n^2\right\}. \tag{4.91}$$

Without loss of generality, we may choose $b=1$.

From (4.62), (4.91) with $b=1$ and the formula of Gauss, we know that the horizontal lift $\phi: M\to H_1^{2m+1}(c)\subset \mathbf{C}_1^{m+1}$ of $z: M\to CH^m(4c)$ satisfies

$$\begin{aligned}
&\phi_{xx} = i\sqrt{-c}\ \mathrm{sech}\left(\sqrt{-c}\,x\right)\phi_x - c\phi,\\
&\tilde\nabla_Y\phi_x = \left(i\sqrt{-c}\ \mathrm{sech}\left(\sqrt{-c}\,x\right)+\sqrt{-c}\tanh\left(\sqrt{-c}\,x\right)\right)Y,\\
&\tilde\nabla_Y\tilde\nabla_Z\phi = \left\{i\sqrt{-c}\ \mathrm{sech}\left(\sqrt{-c}\,x\right)\phi_x - c\phi\right\}\langle Y,Z\rangle + \nabla_Y Z.
\end{aligned} \tag{4.92}$$

Solving the first equation of (4.92) yields

$$\begin{aligned}
\phi = {}& A(x_2,\dots,x_n)h(x)\\
&+B(x_2,\dots,x_n)\cosh\left(\sqrt{-c}x\right)\exp\left\{2i\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\},
\end{aligned} \tag{4.93}$$

for some $\mathbf{C}^{n+1}$-valued vector functions A and B, where

$$\begin{aligned}
h(x) = {}& \frac{\cosh\left(\sqrt{-c}\,x\right)}{2\sqrt{-c}}\exp\left\{2i\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}\,x}{2}\right)\right)\right\}\\
&\times\left\{2\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}\,x}{2}\right)\right)+\mathrm{sech}^2\left(\sqrt{-c}\,x\right)\left(i+\sinh\left(\sqrt{-c}\,x\right)\right)\right\}.
\end{aligned} \tag{4.94}$$

From (4.94) and the second equation of (4.92), we know that A is a constant vector, say c_0.

By applying (4.93), (4.94), the third equation of (4.92) and a long straightforward computation, we obtain

$$\frac{\partial B}{\partial x_j \partial x_k} = i\sqrt{-c}\,\delta_{jk}c_0, \quad i,k = 2,\dots,n. \tag{4.95}$$

Hence, B takes the following form:

$$B(x_2,\dots,x_n) = c_1 + \sum_{j=2}^{n} c_j x_j + \frac{i}{2}c_0\sqrt{-c}\sum_{j=2}^{n} x_j^2, \tag{4.96}$$

for some constant vectors $c_1,\dots,c_n$. Combining (4.93) and (4.96) we obtain

$$\begin{aligned}\phi \;=\; & c_0 h(x) + \left(c_1 + \sum_{j=2}^{n} c_j x_j + \frac{i}{2}c_0\sqrt{-c}\sum_{j=2}^{n} x_j^2\right) \\ & \times \cosh\left(\sqrt{-c}\,x\right)\exp\left\{2i\tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}x}{2}\right)\right)\right\}.\end{aligned} \tag{4.97}$$

By choosing the initial conditions:

$$\phi(0,\dots,0) \;=\; \left(\frac{i}{\sqrt{-c}}, 0,\dots,0\right),$$

$$\begin{aligned}\phi_x(0,\dots,0) \;&=\; (0,1,0,\dots,0),\\ \phi_{x_2}(0,\dots,0) \;&=\; (0,0,1,0,\dots,0),\\ &\;\;\vdots\\ \phi_{x_n}(0,\dots,0) \;&=\; (0,\dots,0,1,0,\dots,0)\end{aligned}$$

we obtain statement (11) in this case.

CASE (γ-d-3). $b^2 + c < 0$.

In this case, Hiepko's result implies that M is locally a warped product of a real line and $H^{n-1}(-1)$ with warped product metric given by

$$g = dx^2 - \frac{\cosh^2\left(\sqrt{-c}\,x\right)}{b^2 + c} g_{-1}, \tag{4.98}$$

where

$$g_{-1} = dx_2^2 + \sinh^2 x_2\left\{dx_3^2 + \cos^2 x_3 dx_4^2 + \cdots + \prod_{k=3}^{n-1}\cos^2 x_k dx_n^2\right\} \tag{4.99}$$

is a metric on $H^{n-1}(-1)$ with constant negative curvature -1.

From (4.62), (4.98), and the formula of Gauss, we know that the horizontal lift $\phi : M \to H_1^{2m+1}(c) \subset \mathbf{C}_1^{m+1}$ of $z : M \to CH^m(4c)$ satisfies

$$\begin{aligned} \phi_{xx} &= ib \operatorname{sech}\left(\sqrt{-c}\, x\right) \phi_x - c\phi, \\ \tilde{\nabla}_Y \phi_x &= \left(ib \operatorname{sech}\left(\sqrt{-c}\, x\right) + \sqrt{-c} \tanh\left(\sqrt{-c}\, x\right)\right) Y, \\ \tilde{\nabla}_Y \tilde{\nabla}_Z \phi &= \left\{ib \operatorname{sech}\left(\sqrt{-c}\, x\right) \phi_x - c\phi\right\} \langle Y, Z\rangle + \nabla_Y Z. \end{aligned} \tag{4.100}$$

Solving the first and second equations of (4.100) yields

$$\begin{aligned} \phi = \;& c_0 \left(ib - \sqrt{-c} \sinh\left(\sqrt{-c}\, x\right)\right) + B(x_2, \ldots, x_n) \cosh\left(\sqrt{-c}\, x\right) \\ & \times \exp\left\{2i \left(\frac{b}{\sqrt{-c}}\right) \tan^{-1}\left(\tanh\left(\frac{\sqrt{-c}\, x}{2}\right)\right)\right\}. \end{aligned} \tag{4.101}$$

From (4.101) and the third equation of (4.100) we conclude that B satisfies

$$\begin{aligned} B = \;& c_1 \cosh x_2 + \sinh x_2 \left(c_2 \sin x_3 + c_3 \cos x_3 \sin x_4 + \cdots \right. \\ & \left. \cdots + c_n \cos x_3 \cdots \cos x_{n-1}\right) \end{aligned} \tag{4.102}$$

Combining (4.101) and (4.102) we obtain

$$\begin{aligned} \phi = \;& c_0 \left(ib - \sqrt{-c} \sinh\left(\sqrt{-c} x\right)\right) \\ & + \left(c_1 \cosh x_2 + \sinh x_2 (c_2 \sin x_3 + c_3 \cos x_3 \sin x_4 + \cdots \right. \\ & \quad \left. \cdots + c_n \cos x_3 \cdots \cos x_{n-1}\right) \\ & \times \cosh\left(\sqrt{-c} x\right) \exp\left\{2i \left(\frac{b}{\sqrt{-c}}\right) \tan^{-1}\left(\tanh\left(\frac{\sqrt{-c} x}{2}\right)\right)\right\}. \end{aligned} \tag{4.103}$$

By choosing the initial conditions:

$$\begin{aligned} \phi(0, \ldots, 0) &= \frac{1}{\sqrt{-(b^2 + c)}} \left(1, 0, \ldots, 0, \frac{ib}{\sqrt{-c}}, 0, \ldots, 0\right), \\ \phi_x(0, \ldots, 0) &= \frac{1}{\sqrt{-(b^2 + c)}} \left(ib, 0, \ldots, 0, -\sqrt{-c}, 0, \ldots, 0\right), \\ \phi_{x_2}(0, \ldots, 0) &= \frac{1}{\sqrt{-(b^2 + c)}} (0, 1, 0, \ldots, 0), \\ &\vdots \\ \phi_{x_n}(0, \ldots, 0) &= \frac{1}{\sqrt{-(b^2 + c)}} (0, \ldots, 0, 1, 0, \ldots, 0), \end{aligned}$$

we obtain statement (12).

Conversely, by straightforward long computations, we can prove that the submanifolds given in statements (1)–(12) are slumbilical. $\square$

REMARK 4.1. The totally real submanifolds given by (4.6) and (4.7) are complex extensors in the sense of [3].

References

[1] B.Y. Chen: Geometry of Slant Submanifolds, Katholieke Universiteit Leuven, 1990.
[2] B.Y. Chen: *Interaction of Legendre curves and Lagrangian submanifolds*, Israel J. Math. **99** (1997), 69–108.
[3] B.Y. Chen: *Complex extensors and Lagrangian submanifolds in complex Euclidean spaces*, Tôhoku Math. J. **49** (1997), 277–297.
[4] B.Y. Chen and K. Ogiue: *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974), 257–266.
[5] B.Y. Chen and K. Ogiue: *Two theorems on Kaehler manifolds*, Michigan Math. J. **21** (1974), 225–229.
[6] B.Y. Chen and Y. Tazawa: *Slant submanifolds of complex projective and complex hyperbolic spaces*, Glasgow Math. J. **42** (2000), 439–454.
[7] F. Dillen and S. Nölker: *Semi-parallelity, muti-rotation surfaces and the helix-property*, J. Reine Angew. Math. **435** (1993), 33–63.
[8] S. Hiepko: *Eine innere Kennzeichung der verzerrten Produkte*, Math. Ann. **241** (1979), 209–215.
[9] K. Kenmotsu and D. Zhou: *The classification of the surfaces with parallel mean curvature vector in two dimensional complex space forms*, Amer. J. Math. **122** (2000), 295–317.
[10] K. Ogiue: *Differential geometry of Kähler submanifolds*, Adv. Math. **13** (1973), 73–113.
[11] H. Reckziegel: *Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion*, Global Diff. Geom and Global Analysis (1984), Lecture Notes in Mathematics, Springer Verlag (1985), 264–279.
[12] Y. Tazawa: *Construction of slant submanifolds*, Bull. Inst. Math. Acad. Sinica, **22** (1994), 153–166; II, Bull. Soc. Math. Belg. (New Series) **1** (1994), 569–576.

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